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Part I reports the somewhat mutually inconsistent treatments of "if–then–" by logic and probability is recounted and used to motivate a formal axiomatic development of conditional propositions in terms of partially-defined, measurable characteristic functions on a sample space. The characteristic function of a conditional proposition $(a|b)$, "a given b", indicates for each instance ω in the sample space whether 1) $(a|b)$ applies and is true for ω, or 2) $(a|b)$ applies amd is false for ω, or 3) $(a|b)$ is inapplicable since b is false. Four 3-valued truth tables (always available by a representation theorem of I. R. Goodman) characterize the "and", "or", "not" and "if–then–" operations of this algebra and capture the third truth state of "inapplicable" for conditional propositions. This leads to an extension of the fundamental theorem of boolean algebra to conditional propositions. Finally, a set of four 4-valued truth tables is offered as a candidate for capturing both the "inapplicable" and "unknown" truth states.

Part II reports some of the key issues giving rise to conditional event algebras. A rigorous formulation of the basic problem is presented together with a listing of natural properties which such conditional event algebras may be expected to satisfy. Most approaches to the issue have treated conditional events as—in effect—as generalized types of boolean functions. A review is presented of the two leading candidate algebras proposed by each of those authors. However, despite a number of desirable properties these enjoy, there are several difficulties that also occur, including formulation of higher order conditioning, modeling of independent information, and formulation of conditional random variables. A new approach is proposed using a countable product space construction in which all of the above issues, and more, are successfully treated. The major drawback in implementing this approach is that calculations increase exponentially in comparison to that enjoyed by the former approaches.

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Conditional event algebras and conditional probability logics

PHILIP G. CALABRESE – I. R. GOODMAN (*)

ABSTRACT. – The paper is in two parts. In Part I, the somewhat mutually inconsistent treatments of "if - then - " by logic and probability is recounted and used to motivate a formal axiomatic development of conditional propositions in terms of partially-defined, measurable characteristic functions on a sample space. The characteristic function of a conditional proposition $(a|b)$, "a given b", indicates for each instance ω in the sample space whether 1) $(a|b)$ applies and is true for ω, or 2) $(a|b)$ applies and is false for ω, or 3) $(a|b)$ is inapplicable since b is false. Four 3-valued truth tables (always available by a representation theorem of I.R. Goodman) characterize the "and", "or", "not" and "if - then - " operations of this algebra and capture the third truth state of "inapplicable" for conditional propositions. This leads to an extension of the fundamental theorem of boolean algebra to conditional propositions. Finally, a set of four 4-valued truth tables is offered as a candidate for capturing both the "inapplicable" and "unknown" truth states.

Part II scopes out some of the key issues giving rise to conditional event algebras. A rigorous formulation of the basic problem is presented together with a listing of natural properties which such conditional event algebras may be expected to satisfy. It is pointed out that most approaches to the issue have treated conditional events as -in effect - as generalized types of boolean functions. A brief review is presented of the two leading candidate algebras proposed by each of those authors. However, despite a number of desirable properties these enjoy, there are several difficulties that also occur, including formulation of higher order conditioning, modeling of independent information, and formulation of conditional random variables. A new approach is proposed using a countable product space construction in which all of the above issues, and more, are successfully treated. The major drawback in implementing this approach is that calculations increase exponentially in comparison to that enjoyed by the former approaches.

KEY WORDS: *Conditional events, conditional propositions, conditional probability, conditional logic, deduction, non-monotonic, artificial intelligence, expert systems, three-valued logic, truth-tables.*

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Part I
An extension of the fundamental theorem of boolean algebra to conditional propositions

Philip G. Calabrese

1. Introduction.

The Treatment of "if - then - " in Logic Versus Probability

Prior to 1933, when A. N. KOLMOGOROV [18] first published his celebrated axioms for probability theory, he had found that the standard treatment of "if - then -" in logic was inconsistent with the laws of probability. Since there was no probabilistically acceptable algebra of "if - then - " (conditional propositions), Kolmogorov simply defined a conditional probability without defining any underlying conditional propositions. Nor did he define boolean-like operations on such conditionals. To this day there remains this remarkable breach between conditional logic and conditional probability with respect to "if - then - " statements: In standard 2-valued logic, conditional statements like "if b then a" are routinely reduced to the statement "either a or not b", the so-called material conditional. For instance, when proving a theorem of the form "if b then a", a mathematician can prove that in all cases "either a is true or b is false". Since mathematical proofs require that there be no exceptions (2-valued logic) this reduction works out fairly well. But as soon as the propositions involved become uncertain, the above reduction can greatly distort the standard probabilistic measure of the partial truth of a conditional statement "if b then a".

In probability theory, "if b then a" has the conditional probability $P(a|b)$, which is just the ratio of the probability of "a and b" to the probability of "b". In symbols, $P(a|b) = P(a \text{ and } b)/P(b)$. But this probability is generally much less[(1)] than the probability of the statement "either a or not b" routinely used in 2-valued logic to reduce "if b then a". The conditional probability can even be close to zero while $P(a \text{ or not } b)$ is close to one. The most extreme situation occurs when the conditional probability is undefined (because the premise "b" has probability zero) while the statement "either

[(1)]The difference is $P(a \text{ or not } b) - P(a|b) = (1-P(b))(1-P(a|b))$ as expressed by P. CALABRESE [3]. So $P(a \text{ or not } b) = p(a|b)$ if and only if either $P(b) = 1$ or $P(a|b) = 1$. For an elaboration see P. CALABRESE [4] and [5], p. 682.

a or not b" is a certainty no matter what the truth of "a" (because "not b" is true). Furthermore, LEWIS [19] actually proved that except for trivial Boolean algebras $\mathcal{B}$, any function or operation f of boolean propositions "a" and "b" with the property that $P(f(a,b)) = P(a|b)$ for all a and b in $\mathcal{B}$ with $P(b) \neq 0$, must have its image, $f(a,b)$, outside the initial boolean algebra $\mathcal{B}$.

Being aware of the above logic-probability breach, various author's (G. BOOLE [2], S. MAZURKIEWICZ [20-22], B. DE FINETTI [9], G. SCHAY [29], T. HAILPERIN [17], E. ADAMS [1], N. NILSSON [25], P. CALABRESE [4-8], I.R. GOODMAN et al [12-16], H.T. NGUYEN and G.S. ROGERS [24] and E.A. WALKER [30]) have attempted to define operations on conditional propositions that are consistent with both logic and probability. These efforts have resulted in several different algebras of conditional propositions and also several ways to represent conditional propositions, an area of research that has recently been called conditional event algebra/conditional probability logic (CEAPL).

Overview. First propositions and conditional propositions are algebraically formulated in ways that are by now standard in the CEAPL literature. Next a representation theorem (due to I. R. Goodman) is proved characterizing all binary operations on conditional propositions in terms of 3-valued truth tables, and conversely. Boolean-like binary operations of "and", "or", "not" and "if - then - " are then defined on the conditional propositions according to P.G. CALABRESE [4-6], and the four 3-valued truth tables for these operations are exhibited. The third value of these 3-valued truth tables captures the the notion of an "inapplicable" conditional proposition, one whose condition is false, rather than a third truth-value of "unknown" for propositions. The subsection "Fundamental Theorem of Boolean Algebra Extended to Conditionals" begins with a definition of a conditional Boolean function and continues with a new result, an extension of the Fundamental Theorem of Boolean Algebra to conditional propositions. The new theorem is then used to prove two corollaries, one of them being Goodman's representation theorem. Finally, a new four-valued logic is suggested in order to capture both the "inapplicable" and "unknown" truth states of conditional propositions.

2. Fundamentals of Conditional Probability Logic.

The formal development of CEAPL can be accomplished most simply in terms of partially defined, measurable indicator (characteristic) functions on a sample space. This approach was early proposed by B. DE FINETTI [9] and later utilized by G. SCHAY [29]. (Also see P. CALABRESE [4], p. 234 and especially [5], pp. 684-686, and [6], pp. 75-82.) Alternatively, the development of CEAPL can be defined in terms of ordered pairs of propositions; still another approach is via algebraic filters. (For the latter developments, see P. CALABRESE [4], pp. 203-214 and I.R. Goodman et al [13], pp. 25-46). First the connection between probabilistic events and logical propositions will be briefly recounted.

Events and Propositions

Let $\mathcal{P} = (\Omega, \mathcal{B}, P)$ be a probability space of individual instances Ω, events (an algebra of subsets of instances) $\mathcal{B}$, and probability measure P. Then the characteristic function of each measurable subset $B, B \in \mathcal{B}$ is a unique measurable indicator function $q_B : \Omega \rightarrow \{0,1\}$ from Ω to the 2-element boolean algebra $\{0,1\}$ defined as follows:

$$q_B(\omega) = \begin{cases} 1, & \text{if} \quad \omega \in B\,, \\ 0, & \text{if} \quad \omega \in B' \end{cases} \tag{1}$$

The function q (dropping the subscript) is a "proposition" in the sense that for each $\omega \in \Omega$, either q is true for ω, meaning $q(\omega) = 1$, or else q is false for ω, meaning $q(\omega) = 0$. L will denote the set of all propositions of $\mathcal{P}$. Conversely, each measurable indicator function q defines a unique measurable subset $B, B \in \mathcal{B}$ by

$$B = q^{-1}(1) = \{\omega \in \Omega : q(\omega) = 1\} \tag{2}$$

B is the measurable subset of cases (instances) for which q is true, and $P(B)$ is the probability measure of the partial truth of q.

In this correspondence between measurable subsets (i.e., probabilistic events) and measurable indicator functions (i.e., propositions) the universe of all possible cases Ω corresponds to the unity indicator function, to those propositions that are true in all cases — necessary and provable. The empty set Φ corresponds to the zero indicator function, to those propositions that are false in all cases — impossible and contradictory. Two propositions p and q are equivalent if and only if they are equal as functions.

Boolean Operations on Propositions. The boolean operations of union ($\cup$), intersection ($\cap$) and complement ($'$) defined on the boolean algebra (or sigma-algebra) $\mathcal{B}$ of events of $\mathcal{P}$ naturally induce boolean operations on the propositions of L: For arbitrary events A and B in $\mathcal{B}$, or propositions p_A, p_B in L, define

$$\begin{aligned} p_A \vee p_B &= p_{(A\cup B)} \\ p_A \wedge p_B &= p_{(A\cap B)} \\ \neg(p_A) &= p_{A'} . \end{aligned} \tag{3}$$

This is equivalent to the equations

$$\begin{aligned} (p \vee q)(\omega) &= p(\omega) \vee q(\omega) \\ (p \wedge q)(\omega) &= p(\omega) \wedge q(\omega) \\ \neg p(\omega) &= \neg(p(\omega)) \end{aligned} \tag{4}$$

where the operations of disjunction ($\vee$), conjunction ($\wedge$ or juxtaposition) and negation ($\neg$) on the right hand side of equations (4) are in the 2-element boolean algebra $\{0,1\}$. $\mathcal{L}$ will denote the boolean algebra of propositions L as generated by the probability space $\mathcal{P}$.

Conditional Events and Conditional Propositions

Consider now that each ordered pair, $(B|A)$ of measurable subsets B, A in $\mathcal{B}$ with corresponding indicator functions q, p, defines a unique *domain-restricted* measurable indicator function $(q|p) : A \to \{0,1\}$, from A to the 2-element boolean algebra as follows:

$$(q|p)(\omega) = \begin{cases} 1, & \text{if} \quad \omega \in (A \cap B), \\ 0, & \text{if} \quad \omega \in (A \cap B'), \\ \text{undefined}, & \text{if} \quad \omega \in A' \end{cases} \tag{5}$$

This can be expressed in terms of the unconditioned propositions p and q by

$$(q|p)(\omega) = \begin{cases} q(\omega), & \text{if} \quad p(\omega) = 1\,, \\ \text{undefined}, & \text{if} \quad p(\omega) = 0 \end{cases} \tag{6}$$

$(q|p)$ is a "conditional proposition" in the sense that if p is true for ω then either $(q|p)$ is true for ω or $(q|p)$ is false for ω. But we say that $(q|p)$ "does not apply" (i.e., is undefined or inapplicable) for those ω for which p is false.

Thus $(q|p)$ has three truth states. $(q|p)$ is simply q, restricted to $p^{-1}(1)$, the subset on which p is true. The set of all conditional propositions of $\mathcal{P}$ will be denoted L/L. Conversely, each domain-restricted, measurable indicator function $(q|p)$ defines a unique conditional event $(B|A)$, where A and B are measurable subsets determined by $A = p^{-1}(1)$ and $B = q^{-1}(1)$. A is the measurable subset on which p is true; B is the measurable subset on which q is true, and $B \cap A$ is the measurable subset on which both q and p are true. For non-zero $P(A)$, $P(B|A) = P(B \cap A)/P(A)$ is the conditional probability of q given p, which is denoted $P(q|p)$.

Definition of Equivalence. Two conditional propositions $(q|p)$ and $(s|r)$ are equivalent, i.e. $(q|p) = (s|r)$, if and only if they are equal as indicator functions, that is, if and only if they have the same domain and are equal on this common domain. Note that it easily follows that two conditionals $(q|p)$ and $(s|r)$ are equivalent if and only if they have equivalent premises and their conclusions are equivalent in conjunction with that premise. That is,

(7) $$(q|p) = (s|r) \quad \text{if and only if} \quad (p = r) \quad \text{and} \quad (qp = sr)\,.$$

Note that if $(q|p)$ is an arbitrary conditional then $(q|p) = (qp|p)$. A conditional proposition $(q|p)$ is said to be in *reduced form* if $q = qp$. It is also easy to see that equivalent conditional propositions are assigned the same conditional probability.

Operations on Conditionals and 3-Valued Logic

It is now time to specify boolean-like operations "and", "or", and "not" on conditional propositions and also the operation of iterated (nested) conditioning.

Representation Theorem for Operations on Conditional Propositions. In this regard, I.R. Goodman ([12] and [13], p. 81) has proved a fundamental representation theorem for operations on conditional propositions. According to this theorem, all operations on conditionals that are made up of boolean antecedents and consequents can be expressed as 3-valued truth tables, and conversely. These truth tables are applied in individual instances ω (models) according to the correspondence

$(a|b)$ is "true" – a and b are both true in ω

$(a|b)$ is "false" – a is true and b is false in ω

$(a|b)$ is "undefined" – b is false in ω

This theorem is an extension of the well-known theorem from Boolean algebra that asserts that any Boolean function of propositions is completely determined by its action on the two propositions 0 and 1. For instance any binary operation (such as $\vee$ or $\wedge$) on Boolean propositions is determined by its action on the 0 and 1 propositions. But in the realm of conditionals, every such function on conditional propositions is completely determined by its action on the three conditional propositions $(0|1)$, $(1|1)$ and $(0|0)$. Conversely, every such function of conditionals generates such a 3-valued truth table. For simplicity, restrict attention to binary operations, that is, to 2-place functions.

Representation Theorem (Goodman). Let f be any 2-place function f on conditionals of the form

$$f((a|b),(c|d)) = (g(a,b,c,d)|h(a,b,c,d)) \tag{8}$$

where g and h are Boolean propositions. Then f defines a 3-valued assignment of the 3 conditional propositions $(1|1),(0|1)$ and $(0|0)$ to themselves. Conversely, any such assignment defines such a 2-place function on conditional propositions.

Proof. - Clearly, when a,b,c and d are in $\{0,1\}$, then $g(a,b,c,d)$ and $h(a,b,c,d)$ are both in $\{0,1\}$ and so $(g(a,b,c,d)|h(a,b,c,d))$ is $(1|1)$, $(0|1)$ or $(0|0)$ since $(1|0) = (0|0)$. So every such 2-place function f on conditionals defines a 3-valued assignment of the 3 conditionals $(1|1),(0|1)$ and $(0|0)$ to themselves. Conversely, any such assignment can be used to define a unique binary function on conditionals: Let $(1|1) = 1, (0|1) = 0$ and $(0|0) = I$ (for undefined or "inapplicable"), and let k be any binary function of the 3 values 1, 0 and I. Then k assigns to each pair of indicator functions $(a|b)$ and $(c|d)$ the indicator function $k(a|b) : \omega \to k((a|b)(\omega),(c|d)(\omega))$, which is measurable because k is a discrete function of the three values 1, 0, and I, and so therefore measurable and because the composite of measurable functions is measurable.

GOODMAN [13], pp. 103-6, has shown that the the three operations of "and", "or" and "not" (designated GNW) derived by himself, H.T. Nguyen and E. A. Walker correspond to the 3-valued truth tables of J. Lukasiewicz, whereas the the author's operations (designated SAC) correspond to the 3-valued truth tables of B. Sobocinski. (See N. RESCHER [27] for an account of these 3-valued logics.)

Boolean-like Operations for Conditionals. Operations on L/L have been defined and motivated in [4-8]. For arbitrary conditionals $(q|p)$ and $(s|r)$ these operations are

(9a) $$(q|p) \vee (s|r) = (qp \vee sr)|(p \vee r)$$

(9b) $$(q|p) \wedge (s|r) = (qp\neg r \vee \neg psr \vee qpsr)|(p \vee r)$$

(9c) $$\neg(q|p) = (\neg qp|p)$$

Briefly, the relative negation $\neg(q|p)$ of $(q|p)$ is just $(\neg q|p)$, which is also $(\neg qp|p)$, since $(q|p) \vee (\neg q|p) = (1|p)$ and $(q|p) \wedge (\neg q|p) = (0|p)$, and $(1|p)$ and $(0|p)$ have conditional probabilities 1 and 0 respectively. These are the reasons for equation (9c). The disjunction $(q|p) \vee (s|r)$ is applicable when either p or r is true. It is applicable and also true when either both q and p are true or both s and r are true. Otherwise, $(q|p) \vee (s|r)$ is applicable and false. This motivates equation (9a). Similarly, $(q|p) \wedge (s|r)$ is applicable when either p or r is true; it is false and applicable when q is false and p is true or when s is false and r is true. Otherwise it is true and applicable. So $(q|p) \wedge (s|r)$ is true and applicable on $\neg(\neg qp \vee \neg sr) = (q \vee \neg p)(s \vee \neg r) = (qp\neg r \vee \neg psr \vee qpsr \vee \neg p\neg r) = qp\neg r \vee \neg psr \vee qpsr$, since $\neg p\neg r = 0$ if $(p \vee r)$ is true, that is, if $(q|p) \wedge (s|r)$ is applicable. This motivates equation (9b).

With the operations of "or" ($\vee$), "and" (juxtaposition or $\wedge$) and "not" ($\neg$), the set L/L of conditional propositions $(q|p)$ includes an isomorphic copy of the original boolean algebra of propositions according to the identification

(10) $$(p|1) \iff p$$

Furthermore, for any fixed non-zero proposition p, the set of conditionals $\{(q|p) : \text{all } q \in L\}$ forms a boolean algebra, which is denoted $\mathcal{L}/p$. However L/L together with these three operations does not form a boolean algebra (although it has many boolean subalgebras). L/L forms a join lattice with respect to $\vee$ and a meet lattice with respect to $\wedge$. Distributivity no longer holds in general. 1 and 0 are no longer absolute units but $(1|0)$ is an absolute unit. Absolute negations do not generally exist and absorption may not hold. (For an elaboration see CALABRESE [4], pp. 226-227 and [6], pp. 93-96 and [34] of Part II.)

The De Morgan formulas can also be proved for conditionals:

(11) $$\neg[(q|p) \vee (s|r)] = \neg(q|p) \wedge \neg(s|r)$$

(12) $$\neg[(q|p) \wedge (s|r)] = \neg(q|p) \vee \neg(s|r)$$

The Conditional Closure

Although the conditioning operator is not a closed operation on propositions, it can be made a closed operation on conditional propositions. This is reminiscent of how the fraction formed by two integers is not in general equal to another integer but the fraction formed by two fractions is again equal to a fraction. In general conditional conditionals are of the form $(q|p)|(s|r)$. The mixed forms $((q|p)|s)$ and $(q|(s|r))$ for propositions q, p, s, and r can be expressed as the special cases $(q|p)|(s|1)$ and $(q|1)|(s|r)$ respectively.

Definition of Iterated Conditionals. Let $(q|p)$ and $(s|r)$ be arbitrary conditionals. Then define the iterated conditional proposition $(q|p)|(s|r)$ by

$$(q|p)|(s|r) = (q|p \wedge (s|r)) \tag{13}$$

This is a generalization of the so-called "import-export" principle that "if c then (if b then a)" is equivalent to "if c and b then a". By equation (9b) equation (13) conveniently reduces to

$$(q|p)|(s|r) = q|(p(s \vee \neg r)) \tag{14}$$

Applying this to the mixed form cases yields

$$((q|p)|s) = (q|ps) \tag{15a}$$

$$(q|(s|r)) = (q|(s \vee \neg r)) \tag{15b}$$

The last formula (15b) shows that *as a condition*, $(s|r)$ is equivalent to $(s \vee \neg r)$. This is one place where the old identification of "if p then q" with "either q or else not p" finds an appropriate place in the new theory. For a purely algebraic justification of equation (14) see [4], pp. 214-219 and [40] of Part II.

The collection L/L of all conditional propositions under the four operations "or" ($\vee$), "and" (juxtaposition or $\wedge$), "not" ($\neg$ or $'$) and "given" ($|$) forms a closed system which the author has termed the *conditional closure* of the boolean logic $\mathcal{L}$, formally denoted $\mathcal{L}/\mathcal{L}$.

False Versus Not True. A good way to view conditional conditionals is by interpreting "c given d" as "c given that d is not false" as contrasted from "c given that d is true". In the boolean case these two are equivalent. But in the realm of conditionals, they are no longer equivalent. Thus "a given that $(c|d)$ is true" is not the same as "a given that $(c|d)$ is not false" due to the third truth state of "inapplicable". "$(c|d)$ is not false" means "$(c|d)$ is true or inapplicable". That is, in terms of the original propositions, it means "either $(c \wedge d)$ is true or d is false". That is, (a given $(c|d)$ is not false) $= (a|c \vee \neg d)$, which agrees with equation (14). In contrast, "$a|(c|d)$ is true" is equivalent to "a given $(c \wedge d)$ is true", which is also equivalent to "$((a|c)|d)$".

Truth Value Representation. As expressed in the proof of the Representation Theorem, by considering all possible assignments of T (true), F (false), and I (inapplicable) to two initial conditional propositions $(q|p)$ and $(s|r)$ and then applying the operations (9a), (9b), (9c) and (14) for "or", "and", "not" and "given" respectively, the following four 3-valued truth tables for conditional propositions are easily generated:

	AND	OR	GIVEN	NOT
	$T\,F\,I$	$T\,F\,I$	$T\,F\,I$	
T	$T\,F\,T$	$T\,T\,T$	$T\,I\,T$	F
F	$F\,F\,F$	$T\,F\,F$	$F\,I\,F$	T
I	$T\,F\,I$	$T\,F\,I$	$I\,I\,I$	I

Note here that "not true" means "false or inapplicable", that "not false" means "true or inapplicable", and that "true" and "false" are no longer opposites.

Fundamental Theorem of Boolean Algebra Extended to Conditionals

Inspired by Goodman's Representation Theorem, the author was led to investigate an extension of the Fundamental Theorem of Boolean algebra to conditional propositions. Recall (see, for instance, P. ROSENBLOOM [28], p. 5) that a Boolean function of one variable is a function that can be formed by starting with constant functions and the identity function and applying to these the operations of conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$). This definition can easily be extended to functions of any finite number of variables. For Boolean functions f of one variable x the fundamental theorem of Boolean algebra states that

(16) $$f(x) = (f(1) \wedge x) \vee (f(0) \wedge x'),$$

where x' has been written for $\neg x$ to condense notation. It is then a corollary that any such Boolean function is completely determined by its action on the two propositions 1 and 0. For Boolean functions of two variables, this becomes

$$(17) \qquad f(x,y) = f(1,1)xy \vee f(1,0)xy' \vee f(0,1)x'y \vee f(0,0)x'y'$$

where juxtaposition has replaced $\wedge$. The above functions are said to be expressed in "disjunctive normal form".

Definition of Conditional Boolean Functions. A conditional Boolean function of one conditional variable $(x|y)$ is a function $f : L/L \to L/L$ given by

$$(18) \qquad f(x|y) = (g(x,y)|h(x,y))$$

where g and h are Boolean functions of two Boolean propositional variables x and y. Similarly, a conditional Boolean function of two conditional variables is a function f given by

$$(19) \qquad f((x|y),(w|z)) = (g(x,y,w,z)|h(x,y,w,z))$$

where g and h are Boolean functions of four Boolean propositional variables. f is a conditional Boolean function of n conditional variables if

$$(20) \qquad f(x_1|y_1, x_2|y_2, ..., x_n|y_n) = (w|z)$$

where $w = g(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ and $z = h(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ are Boolean functions of $2n$ variables.

Fundamental Theorem of Boolean Algebra Extended to Conditionals. If f is a conditional Boolean function of one variable then f can be uniquely expressed by

$$(21) \qquad f(x|y) = (f(1|1)|xy) \vee (f(0|1)|x'y) \vee (f(0|0)|y').$$

This expression for $f(x|y)$ will be called the disjunctive normal form.

LEMMA. - If a, b, c, and d are any four propositions then

$$(22) \qquad (a \vee b)|(c \vee d) = (a|c) \vee (a|d) \vee (b|c) \vee (b|d)$$

Proof of Lemma - $(a \vee b)|(c \vee d) = (a|(c \vee d)) \vee (b|(c \vee d))$ $= (a|c) \vee (a|d) \vee (b|c) \vee (b|d)$

This lemma can easily be extended to any two finite sets of propositions $\{a_i : i = 1, 2, ..., n\}$ and $\{b_j : i = 1, 2, ..., m\}$:

$$(a_1 \vee a_2 \vee \ldots \vee a_n)|(b_1 \vee b_2 \vee \ldots \vee b_m) = \bigvee_{i,j} (a_i|b_j) \tag{23}$$

Proof of Fundamental Theorem - Since f must be a well-defined function, and since $(x|y) = (xy|y)$, it follows that $f(x|y) = f(xy|y) = g(xy, y)|h(xy, y)$. Now, applying the fundamental theorem of Boolean algebra to g yields

$$\begin{aligned} g(xy, y) &= g(1,1)(xy)y \vee g(1,0)(xy)y' \vee g(0,1)(xy)'y \vee g(0,0)(xy)'y' \\ &= g(1,1)xy \vee g(1,0)(0) \vee g(0,1)x'y \vee g(0,0)y' \\ &= g(1,1)xy \vee g(0,1)x'y \vee g(0,0)y' \, . \end{aligned} \tag{24}$$

Similarly, $h(xy, y) = h(1,1)xy \vee h(0,1)x'y \vee h(0,0)y'$. Therefore, using the lemma

$$\begin{aligned} f(x|y) &= [g(1,1)xy \vee g(0,1)x'y \vee g(0,0)y']|[h(1,1)xy \vee h(0,1)x'y \vee h(0,0)y'] \\ &= [g(1,1)xy|h(1,1)xy] \vee [g(1,1)xy|h(0,1)x'y] \vee [g(1,1)xy|h(0,0)y'] \\ &\vee [g(0,1)x'y|h(1,1)xy] \vee [g(0,1)x'y|h(0,1)x'y] \vee [g(0,1)x'y|h(0,0)y'] \\ &\vee [g(0,0)y'|h(1,1)xy] \vee [g(0,0)y'|h(0,1)x'y] \vee [g(0,0)y'|h(0,0)y']. \end{aligned} \tag{25}$$

Now only one of the three conditionals with antecedent $h(1,1)xy$ is nonzero. The others are equivalent to $[0|h(1,1)xy]$ because both $(x'y)(xy) = 0$ and $(y')(xy) = 0$. Similarly for the conditionals with antecedents $h(0,1)x'y$ and $h(0,0)y'$. Thus

$$\begin{aligned} f(x|y) &= [g(1,1)xy|h(1,1)xy] \vee [g(0,1)x'y|h(0,1)x'y] \vee [g(0,0)y'|h(0,0)y'] \\ &= [g(1,1)|h(1,1)xy] \vee [g(0,1)|h(0,1)x'y] \vee [g(0,0)|h(0,0)y'] \\ &= [(g(1,1)|h(1,1))|xy] \vee [(g(0,1)|h(0,1))|x'y] \vee [(g(0,0)|h(0,0))|y'] \\ &= [f(1|1)|xy] \vee [f(0|1)|x'y] \vee [f(0|0)|y'] \, . \end{aligned} \tag{26}$$

To show uniqueness, suppose f and j are two conditional Boolean functions of $(x|y)$. Then f and j can be expressed by

$$f(x|y) = [g(1,1)xy \vee g(0,1)x'y \vee g(0,0)y']|[h(1,1)xy \vee h(0,1)x'y \vee h(0,0)y']$$

$$j(x|y) = [k(1,1)xy \vee k(0,1)x'y \vee k(0,0)y']|[m(1,1)xy \vee m(0,1)x'y \vee m(0,0)y']$$

where g, h, k and m are Boolean functions. Now if the right-hand sides of the above equations are equal, then by the definition of equivalent conditionals,

$$h(1,1)xy \vee h(0,1)x'y \vee h(0,0)y' = [m(1,1)xy \vee m(0,1)x'y \vee m(0,0)y'] \tag{27}$$

and

$$\begin{aligned} &g(1,1)h(1,1)xy \vee g(0,1)h(0,1)x'y \vee g(0,0)h(0,0)y' \\ &= k(1,1)m(1,1)xy \vee k(0,1)m(0,1)x'y \vee k(0,0)m(0,0)y' \end{aligned} \tag{28}$$

Clearly, from the first equation $h(1,1) = m(1,1), h(0,1) = m(0,1)$, and $h(0,0) = m(0,0)$. Therefore the second equation becomes

$$\begin{aligned} &g(1,1)h(1,1)xy \vee g(0,1)h(0,1)x'y \vee g(0,0)h(0,0)y' \\ &= k(1,1)h(1,1)xy \vee k(0,1)h(0,1)x'y \vee k(0,0)h(0,0)y'. \end{aligned} \tag{29}$$

Furthermore, using the uniqueness for the Boolean case of the Fundamental Theorem or directly,

$$\begin{aligned} g(1,1)h(1,1) &= k(1,1)h(1,1) \\ g(0,1)h(0,1) &= k(0,1)h(0,1) \\ g(0,0)h(0,0) &= k(0,0)h(0,0). \end{aligned} \tag{30}$$

Therefore, $[g(1,1)|h(1,1)] = [k(1,1)|m(1,1)]$, and so $f(1|1) = j(1|1)$. Similarly, $f(0|1) = j(0|1)$ and $f(0|0) = j(0|0)$. Therefore

$$\begin{aligned} f(x|y) &= (f(1|1)|xy) \vee (f(0|1)|x'y) \vee (f(0|0)|y') \\ &= (j(1|1)|xy) \vee (j(0|1)|x'y) \vee (j(0|0)|y') = j(x|y)\,. \end{aligned} \tag{31}$$

This completes the proof of the Fundamental Theorem.

The extension to conditional Boolean functions of two or more variables is straightforward but messy to write. For a function f of two conditional variables there are nine disjoined terms. Writing $f((x|y),(w|z))$ as $f(x|y, w|z)$ to reduce parentheses the result is

$$\begin{aligned} f(x|y,w|z) = {} & [f(1|1,1|1)|xywz] \vee [f(1|1,0|1)|xyw'z] \vee [f(1|1,0|0)|xyz'] \\ & \vee [f(0|1,1|1)|x'ywz] \vee [f(0|1,0|1)|x'yw'z] \vee [f(0|1,0|0)|x'yz'] \\ & \vee [f(0|0,1|1)|y'wz] \vee [f(0|0,0|1)|y'w'z] \vee [f(0|0,0|0)|y'z']. \end{aligned} \tag{32}$$

It is now possible to prove Goodman's representation theorem as a corollary to this extended fundamental theorem, where again for simplicity, consider conditional Boolean functions of just one variable.

COROLLARY 1 (Goodman). – *A conditional Boolean function f of one variable induces a unique 3-valued, 2-place truth table on $\mathcal{L}/\mathcal{L}$. Conversely, each such 3-valued, 2-place truth table induces a unique conditional Boolean function of one variable.*

COROLLARY 2 (Goodman). – *There are 3^3 different conditional Boolean functions of one variable; there are 3^{3^n} different conditional Boolean functions of n variables.*

Proof of Corollary 1 - Given a conditional Boolean function f of one variable define the truth table function t by $t(x|y) = f(x|y)$ for $(x|y) \in \{(1|1),(0|1),(0|0)\}$. Conversely, if $t(x|y)$ is a 3-valued truth table function with domain values $(1|1), (0|1)$, or $(0|0)$, then define the conditional Boolean function f by:

$$f(x|y) = [t(1|1)|xy] \vee [t(0|1)|x'y] \vee [t(0|0)|y'] \tag{33}$$

By the uniqueness part of the fundamental theorem, different truth table functions t induce different conditional Boolean functions f.

Proof of Corollary 2 - By the fundamental theorem, f can be uniquely expressed in disjunctive normal form and so there are three possible assignments to each of the three domain elements $\{(1|1),(0|1),(0|0)\}$. Thus there are 3^3 different possible assignments and that number of possible conditional Boolean functions of one variable. For conditional Boolean functions of two variables there are 3^2 different domain elements each of which can be assigned any one of 3 different values. So there are 3^{3^2} different possible assignments and that number of different conditional Boolean functions of two variables. The generalization to n values is straightforward.

Four-Valued Logic and More

It is interesting to note that in their excellent comparison of alternate operations on conditionals, DUBOIS and PRADE ([10], pp. 30-35; [11], pp. 126-131) several times describe $(a|b)$ when b is false as "inapplicable", although they also adopt the interpretation of the truth status in this situation as being "any truth value in $\{0,1\}$." But it seems to this author that an "inapplicable" conditional should be neither true nor false, neither $\emptyset$ nor 1. The truth state described by "either true or false" seems more properly to correspond to the situation when "b" or "a" (or both) remain *unknown*. In this regard, it seems that the GNW-Lukasiewicz operations are more appropriate for modeling "unknown" than for modeling "inapplicable" whereas the SAC (Schay-Adams-Calabrese) trio of operations seems to be more appropriate for modeling "inapplicable" than "unknown". The "unknown" truth state of the GNW-Lukasiewicz system seems well-designed for *unconditional* classical propositions but the "inapplicable" truth state of the SAC-Sobocinski system is designed for *conditional* propositions. With these interpretations, $(a|b)$ should perhaps have four possible truth values or "states" in a more complete formulation of conditional information:

$(a\|b)$ is true	(both a and b are true)
$(a\|b)$ is false	(b is true and a is false)
$(a\|b)$ is inapplicable	(b is false)
$(a\|b)$ is unknown	(b is unknown or b is not false & a is unknown)

(Furthermore, it seems clear that in practice one needs to further qualify these four truth states by saying "presently" before each of the above possibilities because it is essential to be able to update information with additional or even contrary information. Thus, in practice, these four states should also be indexed by time.)

The fact that all binary operations on conditionals that are made up of Boolean antecedents and consequents can be expressed as 3-valued truth tables, and conversely, suggests that any operation on conditionals is appropriately expressed as a 3-valued truth table. However, if both "inapplicable" and "unknown" are to be included in a combined logic then there must be interactions between these two truth states not captured by 3-valued indicator functions. This suggests the following four 4-valued truth tables:

	AND	OR	GIVEN	NOT
	$TFIU$	$TFIU$	$TFIU$	
T	$TFTU$	$TTTT$	$TITU$	F
F	$FFFF$	$TFFU$	$FIFU$	T
I	$TFIU$	$TFIU$	$IIIU$	I
U	$UFUU$	$TUUU$	$UIUU$	U

Note here that the truth value I retains its neutrality: $I \wedge p = I$ and $I \vee p = p$, for p having any truth value in $\{T, F, I, U\}$. Similarly, the truth value U still has that $U \wedge p = U$ unless p is false, and $U \vee p = U$ unless p is true.

The above tables suggest that four-valued indicator functions of some sort may be appropriate for modeling conditional propositions that are subject to being inapplicable as well as possibly having an "unknown" truth state. For example, in the first experiment of rolling a die, the conditions "less than 4" and "not less than 4" may sometimes remain unknown after the experiment. This is realistic in practical applications because sometimes expected information is missing. Four-valued indicator functions could be measurable, real set-valued functions partially defined on the sample space Ω and having truth values of $\{1\}$ (true), $\{0\}$ (false), Φ (inapplicable-undefined) or $\{0,1\}$ (unknown, i.e., both 0 and 1). How these four truth values might be expressed algebraically in terms of operations is an unexplored question.

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REFERENCES

[1] Adams, E.W. (1975): *The Logic of Conditionals: An Application of Probability to Deductive Logic*, Reidel, Boston.

[2] Boole, G. (1940): *An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, 2nd. ed., Open Court; 1st. ed., Macmillan, 1854.

[3] Calabrese, P.G. (1975): The Probability that p Implies q (preliminary report), *Notices American Mathematical Society* **22** (3), A430-A431.

[4] Calabrese, P.G. (1987): An Algebraic Synthesis of the Foundations of Logic and Probability, *Information Sciences* **42**, 187-237.

[5] Calabrese, P.G. (1990): Reasoning with Uncertainty Using Conditional Logic and Probability, in: *Proc. First International Symposium on Uncertainty Modeling and Analysis* by IEEE Computer Society, 682-688.

[6] Calabrese, P.G. (1991): Deduction and Inference Using Conditional Logic and Probability, in: I.R. Goodman, M.M. Gupta, H.T. Nguyen and G.S. Rogers, eds., *Conditional Logic in Expert Systems*, North-Holland, Amsterdam, 71-100.

[7] Calabrese, P.G.: Deductive Systems of Conditional Propositions in Conditional Probability Logic, To be submitted for publication.

[8] Calabrese, P.G.: A Theory of Conditional Information with Applications, to appear in *IEEE Transactions on Systems, Man and Cybernetics*, The Institute of Electrical and Electronics Engineers, Inc.

[9] de Finetti, B. (1972): *Probability, Induction and Statistics*, Wiley, New York.

[10] Dubois, D. and Prade, H. (1990): The logical View of Conditioning and its Application to Possibility and Evidence Theories, *International Journal of Approximate Reasoning* **4**, 23-26.

[11] Dubois, D. and Prade, H. (1991): Conditioning, Non-monotonic Logic, and Non-standard Uncertainty Models, in: I.R. Goodman, M.M. Gupta, H.T. Nguyen and G.S. Rogers, eds., *Conditional Logic in Expert Systems*, North-Holland, Amsterdam, 115-158.

[12] Goodman, I.R. (1990): Three-valued logics and Conditional Event Algebras, in: *Proc. First International Symposium on Uncertainty Modeling and Analysis* (IEEE Computer Society) 31-37.

[13] Goodman, I.R., Nguyen, H.T. and Walker, E.A. (1991): *Conditional Inference and Logic for Intelligent Systems: a Theory of Measure-free Conditioning*, North-Holland, Amsterdam.

[14] Goodman, I.R. (1991): Algebraic and Probabilistic Bases for Fuzzy Sets and the Development of Fuzzy Conditioning, in: I.R. Goodman, M.M. Gupta, H. T. Nguyen and G.S. Rogers, eds., *Conditional Logic in Expert Systems* North-Holland, Amsterdam, 1- 69.

[15] Goodman, I.R. (1991): Evaluation of Combinations of Conditioned Information: A History, *Information Sciences* **57-58**, 79-110.

[16] Goodman, I.R., Gupta, M.M., Nguyen, H.T. and Rogers,G.S. eds. (1991): *Conditional Logic in Expert Systems*, North-Holland, Amsterdam.

[17] Hailperin, T. (1986): *Boole's Logic and Probability*, 2nd ed., North-Holland, Amsterdam.

[18] Kolmogorov, A.N. (1956): *Foundations of the Theory of Probability*, Chelsea; 1st edition: *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933.

[19] Lewis, D. (1976): Probabilities of Conditionals and Conditional Probabilities, *The Philos. Rev.* **85 (3)**, 297-315.

[20] Mazurkiewicz, S. (1932): Zur Axiomatik der Wahrscheinlichkeitslehre, *Comptes rendus des séances de la Société des sciences et des Lettres de Varsovie,* Classe III, Vol **25**, 1-4.

[21] Mazurkiewicz, S. (1934): Über die Grundlagen der Wahrscheinlichkeitsrechnung, *Monatshefte für Mathematik und Physik*, Vol **41**, 343-352.

[22] Mazurkiewicz, S. (1956): Podstawy Rachunku Prawdopodobienstwa (The Foundations of the Calculus of Probability), *Monografie Matematyczne*, Vol. **32**, Panstwowe Wydawnietwo Naukowe, Warszawa.

[23] Nguyen, H.T. (1990): Algebraic Structures of Conditioning Reasoning, in: *Proc. First International Symposium on Uncertainty Modeling and Analysis* (IEEE Computer Society) 564-566.

[24] Nguyen, H.T. and Rogers, G.S. (1991): Conditioning Operators in a Logic of Conditionals, in: I.R. Goodman, M.M. Gupta, H.T. Nguyen and G.S. Rogers, eds., *Conditional Logic in Expert Systems* North-Holland, Amsterdam, 159-179.

[25] Nilsson, N.J. (1986): Probabilistic Logic, *Artificial Intelligence* **28** (**1**), 71-87.

[26] Pearl, J. (1988): *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*, Morgan Kaufman, San Mateo.

[27] Rescher, N. (1969): *Many-valued Logics*, McGraw-Hill.

[28] Rosenbloom, P. (1950): *The Elements of Mathematical Logic*, Dover.

[29] Schay, G. (1968): An Algebra of Conditional Events, *Journal of Mathematical Analysis and Applications* **24**, 334-344.

[30] Walker, E.A. (1991): A Simple Look at Conditional Events, in: I.R. Goodman, M.M. Gupta, H.T. Nguyen and G.S. Rogers, eds.,*Conditional Logic in Expert Systems* North-Holland, Amsterdam, 101-114.

Part II

Basic formulations and a product space approach to conditional events

I.R. Goodman

0. Generalities and Motivations.

Conditional event algebra was developed in response to the actual need - whether perceived or not by the present probability and AI communities - to have an algebraic basis to represent (partial or fully) causal or conditional expressions, in an uncertainty context, which are compatible in a natural way with conditional probabilities. The work initially focused on the representation of such conditional events as one of three demonstrated equivalent forms: intervals of events (and therefore treatable via interval algebra); principal ideal cosets, extending the usual boolean quotient or residue algebras to include the case of non-identical generators or antecedents; three-valued indicator functions (and therefore connections with three- valued logics), extending the usual two-valued ones.

Although many desirable properties were derived for these entities, based upon the development of a particular choice of operators, utilizing functional image extensions of the corresponding classical boolean ones to interval (or coset) form (see references in text below), a growing number of conceptual difficulties has arisen with this approach and those proposed by others. These include: the issues of higher order, or nested, conditioning, conditioning of random variables and relations to conditional events, and connections with classical statistical independence properties and statistical estimation and decision theory, in general. As a consequence of this, another avenue to conditioning has been opened up, based upon a countably infinite product space construction. Although this has led to a more complicated computational structure for determining logical combinations of conditionals and their associated probability evaluations, nevertheless it has also yielded much more satisfactory theoretical foundations for the representation of conditioning.

Many open issues within the new product space approach remain, including: the total imbedding of the originally proposed conditional event operations for the interval approach into the product space setting; the determination of the boolean subalgebra generated by the action of all finite logical operations upon the base conditional events; the full characterization of the various types of product space- derived deduction - this problem turn out to be most efficiently represented in powerdomain terms; the determination of the underlying logic of the product space operations - no longer a simple truth-functional three-valued logic , as in the case of the interval of events approach; application of the new approach to modeling of linguistic information, both indicative and modal, and a cohesive approach to combining of evidence. Promising directions toward solutions to some of these problems may well lie in extending the related work of the logicians Bas Van Fraasen and Van McGee.

This work attempts to determine a unified and computationally feasible theory of causal or implicative forms, compatible with all corresponding conditional probability evaluations, which can be of use in deriving a more mathematically universal view of data fusion and related problems. In this situation, both stochastic and linguistic- based evidence may be present. Two basic motivating examples for carrying out such a scheme are as follows:

a) Partial Modus Ponens Use in Expert Systems, When Validity of Inference Rules is Measured by Corresponding Conditional Probabilities

In expert (or rule-based, or, intelligent) systems, the dominating feature is a collection of relevant inference rules of the form "if b, then a", "if d then c", "if b, then e", etc, When (in effect, by suppressing or incorporating any antecedental information) unconditional data is observed, say b, then all such rules matching b in their antecedents, by modus ponens, fire their consequents via conjunctions as $ab =$ (if b, then a) $\cdot a$, $eb =$ (if b, then e) $\cdot b$, etc., which in turn immediately yield the deducts a, e, etc. In turn, these various outputs $a, e, \ldots$ fire similarly the consequents of all inference rules present which have correspondingly individually $a, e, \ldots$ as their antecedents, etc.

However, in the real world, such perfect matches need not hold and common sense states that each "close" match should produce "close" changes in the consequents. But how much should these be and what mathematical guidelines can we appeal to? Of course, one could try to build in the antecedents and consequents such nuances and changes and still employ traditional modus ponens to fire the rules. But the possibility of establishing a theory of conditionals would allow for the replacement of modus ponens by some nontrivial computation for the conjunctions (if b, then a) $\cdot(b_1)$, (if d, then c) $\cdot(c_1)$, etc., where b_1 is not $= b, c_1$ is not $= c, \ldots$, in general, but are prescreened for proximity to the corresponding antecedents. This can be done, e.g., by use of the natural pseudometric induced by any given probability measure P, $\text{dist}(P)(b,(b_1)) = P((b-(b_1)) \vee ((b_1)-b))$, based on the latter being below a given threshold of probability. In turn, with the development of a suitable algebra of conditionals or "conditional event algebra", one could obtain in a normal boolean or related way computable appropriately determined events (or products of events, etc) A and B, dependent upon (in the first case) a, b, b_1, but not dependent upon any specific probability measure to be applied. This is analogous to the use of ordinary boolean algebra in representing and logically manipulating events prior to any probability evaluation - leading to validity level $P(A|B)$, etc. In the above, note that the antecedents b of inference rules "if b, then a" are generally nontrivial, while those of the data or unconditional events b_1 are tacitly understood to be the universal event L, unless an explicit specified prior is acknowledged.

Thus, in general here we are considering the logical combination of conditionals with nonmatching antecedents.

b) Use of Conditionals in Databases

A second generic potential application of conditional event algebra is simply the representing and logical combining of potential or partial causal relations arising in databases. For example, one may wish to determine first the validity of two possible pairs of database descriptions, where each description consists of those population elements satisfying a particular set of characteristics. The underlying population may be the set of humans in a particular city over 21 years of age who have been surveyed relative to a number of characteristics - with not all individuals necessarily responding and those responding being possibly in error according to some probability distribution. The characteristics may typically include age, height, scaled health level, hair color, etc. Then, in this case, events a, b, c, d could be: a corresponds to over 30 yrs, but under 47 and having blond or brown hair, b corresponds to over 25, having blond or white hair and at health level 5 (on a scale from 1 to 10), with c and d similarly defined. Thus the causality level between b and a and between d and c are naturally measured by the corresponding conditional probabilities $P(a|b)$, $P(c|d)$, respectively. In turn, the conjunction of the causality relations - or in a related vein, the jointness of the causality relations - (if b, then a) and (if d, then c) - may be sought, i.e., P ((if b, then a) $\cdot$ (if d, then c)), where the evaluations hold:

$$P(\text{if } b, \text{ then } a) = P(a|b), \qquad P(\text{if } d, \text{ then } c) = P(c|d).$$

Again, as in example a), we are in general considering the logical combination of conditionals with non-identical antecedents . (When antecedents are identical, or information is completely separate or independent, standard probability techniques are available, as usual.) Extensions of both inner and outer concatenations and other database operations also appear feasible to carry out within this context.

For additional applications of conditional event algebra to both military and non- military uses, including bayesian updating of conditional information and conditional belief construction, see [36]. For yet further motivations, for the introducing of conditional event algebra, see [27]. Preliminary applications of conditional event algebra to default logic and to a wide variety of related problems, including the "penguin triangle" , Simpson's paradox (see especially [4]) and"Poole's paradox " are encouraging [28].

1. The Basic Problem.

Extensive literature search has revealed that at present there is a disconnect between the use of traditional logical methods in treating causal, conditional, or "if- then", information and traditional probabilistic/stochastic approaches to the same information. This issue's origin appears to begin with the original rejection of Boole's proposed full analogues for set/event operations via corresponding elementary arithmetic operations by some of the leading scientists of that period [15]. The omission of an analogue to division to represent conditioning in probability continues into the present, with only a handful of researchers ever considering this long- overlooked problem. That this division of events cannot be represented by simple means was indeed pointed out by CALABRESE [1] and earlier, independently, by POPPER [37], both showing the basic inconsistency between what is taken as the standard classical logical interpretation for implication - the material conditional , which in boolean form for "if b, then a" becomes $b' \vee a (= b' \vee ab)$ and conditional probability:

$$P(b' \vee a) = 1 - P(ab) + P(b') = P(a|b) + (P(b') \cdot P(a'|b)) > P(a|b)\,,$$

in general, except for trivial boundary cases when only equality hold. Calabrese went further and showed that in fact no binary boolean operation could play the role of the argument within conditional probability [2]. But, Lewis, independent of Calabrese and Popper, showed an even more far-reaching result, as detailed below. (See [14], Chapters 0 and 1 for a detailed history of the problem, where the works of de Finetti, Popper, Reichenbach, Mazurkiewicz, Domotor, Copeland, Hailperin, Schay, Adams, and other pioneers in the field are surveyed, as well as the more recent work of Calabrese, Goodman & Nguyen, Dubois & Prade, Scozzafava, and others.)

Lewis' triviality result [33] has added "fuel to the fire", in that it is seen that it is impossible, in general, to obtain a binary operation $g : B \times B$ to B, for B boolean such that for any probability measure P over B and all a, b in B, with $P(b) > 0$,

$$P(g(a,b)) = P(a|b)\,.$$

This has sparked a large number of responses from the logical community to overreact to this constraint by abandoning the search for such possible expressions or conditional events $g(a, b)$ - which we will write from now on as $(a|b)$ - satisfying the above equation. (See, e.g., the various papers in the edited works in [31,32].) However, the above does not preclude the search for such entities outside of B itself (as opposed to the earlier futile work of Copeland and others unaware of Lewis' result - having preceded Lewis' 1976 paper by twenty years and more - see [5,6]).

2. Rigorous Formulation of the Basic Problem.

Basic Mathematical Problem: Given any measurable space (L, B) (B a boolean or sigma algebra with L as its universal element), find another space C having an algebraic structure with operations upon it extending those over B - and labeled for convenience as formally the same - and a binary mapping $(\cdot|\cdot\cdot) : B \times B \to C$, such that using the notation $(a|b)$ for $(\cdot|\cdot\cdot)(a, b)$, for any a, b in B the following conditions hold:

Q1) $(a|L)$ can be identified with a, for all a in B, so that $(\cdot|L) : B \to C$ is an imbedding.

Q2) $(a|b) = (ab|b)$, all a, b in B, dependency of the conditional through only the antecedent and its conjunction with the consequent.

Q3) For any probability measure P over B, there is a function $P_0 : C \to$ unit interval $[0, 1]$, such that $P_0((a|b)) = P(a|b)(= P(ab)/P(b))$, all a, b in B, $P(b) > 0$ - the Stalnaker Thesis (due, in part, to this named individual proposing the above relation which Lewis "destroyed" [39]).

In the above, call any $(a|b)$ "the conditional event with antecedent b and consequent a", or simply "if b, then (possibly) a" or "(possibly) a, given b", or even "b (partially) causes a".

3. Additional Natural Properties Conditional Events Should Possess.

In addition to the three fundamental properties Q1) - Q3) required above, it appears reasonable to also require the following nine additional constraints for conditional events to satisfy:

Q4) $(a|b) \cdot (c|b) = (ac|b)$, $(a|b) \vee (c|b) = (a \vee c|b)$, $(a|b)' = (a'|b)$, all a, b, c in B, standard fixed antecedent homomorphism/ coset-like properties.

Q5) $(a|b) \cdot b = ab$, modus ponens, and more generally, the chaining relation holds $(ac|b) = (a|c) \cdot (c|b)$, for all a, b, c in B.

Q6) $P(a|b) = 0$, for all probability measures P over B , iff $ab = \emptyset$: all a, b in B, for $P(b) > 0$; $P(a|b) = 1$, for all probability measures P over B, iff $ab = b$ iff $b \leq a$; all a, b in B, $P(b) >$ '0, zero-unity consistency properties.

Q7) $P(a|b) = P(c|d)$, for all probability measures P over B, $P(b), P(d) > 0$, iff $(a|b) = (c|d)$; all a, b, c, d in B, the identity consistency with probability.

Q8) $P(a|b) \leq P(c|d)$, for all probability measures P over B, $P(b), P(d) > 0$, iff $(a|b) \leq (c|d)$, for any a, b, c, d in B.

The last relation $\leq$ is defined over the conditional events generated from B by any of the equivalent standard ways a lattice order is determined, if a lattice structure holds; otherwise, some sort of semi-lattice compatibility is required through either $\cdot$ or $\vee$. This property is a conditional probability order consistency one.

Note that from [14], sect. 2.2, Q7) and Q8) reduce to:

Q7′) $(a|b) = (c|d)$ iff $ab = cd$ and $b = d$, provided neither conditional is a zero or unity type.

Q8′) $(a|b) \leq (c|d)$ iff $ab \leq cd$ and $c'd \leq a'b$, provided neither conditional is a zero or unity type.

Q9) If P is any given probability measure over B and a, b, c, d in B are arbitrary such that pairwise (ab, b) and (cd, d) are P-independent, then, provided $P(b), P(d) > 0$,

$$P_0((a|b) \cdot (c|d)) = P(a|b) \cdot P(c|d) .$$

If $bd = 0$, i.e. b and d are disjoint, then the above equation should hold for all probability measures P over B simultaneously, a strong P-independence consistency property.

Q10) For any measurable space (L, B), any positive integer k, denoting the real k-dimensional Borel field over k-dimensional euclidean space R^k as B^k, suppose $(\cdot|\cdot\cdot) : B \times B \to C$ and $(\cdot|\cdot\cdot) : B^k \times B^k \to D^k$ are solutions to the Basic Problem [using for convenience the same conditional event notation, when there is no ambiguity] which yield either boolean or sigma-algebra structures for C and D^k. Then, for any two rv (random variables) $Y : (L, B) \to (R^m, B^m)$ and $Z : (L, B) \to (R^n, B^n)$, there is a rv $W : (L|L), C) \to ((R^{(m+n)}|R^{(m+n)}), D^{(m+n)})$ such that W induces all conditional probabilities simultaneously, i.e., $W^{-1}(a \times b|R^m \times b) = (Y^{-1}(a)|Z^{-1}(b))$, for all a in B^m and all b in B^n. Hence, for all probability measures P over B, $P_0(W^{-1}(a \times b|R^m \times b)) = P(Y^{-1}(a)|Z^{-1}(b))$.

Q11) The results in Q10) are extendable in a natural way to any finite collection of rv's $Y_i : (L, B) \to (R_i^m, B_i^m)$, $Z_i : (L, B) \to (R_i^n, B_i^n)$, such that, assuming cartesian products are extendable to conditional events, for all a_i in B_i^m and b_i in B_i^n, $i = 1, \ldots, r$, for any positive integer r, $W^{-1}[(a_1 \times b_1|R_1^m \times b_1) \times \ldots \times (a_r \times b_r|R_r^m \times b_r)] = (Y_1^{-1}(a_1)|Z_1^{-1}(b_1)) \ldots (Y_r^{-1}(a_r)|Z_r^{-1}(b_r))$ (conjunction), whence, for any probability measure P over B,

$$P_0[W^{-1}[(a_1 \times b_1|R_1^m \times b_1) \times \ldots \times (a_r \times b_r|R_r^m \times (b_r)]] =$$
$$= P_0[(Y_1^{-1}((a_1)|Z_1^{-1}(b_1)) \ldots (Y_r^{-1}(a_r)|Z_r^{-1}(b_r))],$$

probabilities through the hypothesis of the algebraic structure of C-here being at least boolean.

The above properties expressed by Q10) and Q11) can be considered as explicit necessary conditions for extendibility of standard decision theory to a decision theory based upon conditionals. Finally, it should be noted that the change of domain from that of (L, B) for the initial rv Y and Z to that for W is necessary, since it can be shown (see [22], sect. 3.2) that any solution W to Q10) necessary cannot have as its domain the original space (L, B). If such W satisfying Q10) and Q11) exist, it is natural to identify them as "conditional" rv $(Y|Z)$ over their appropriate domains.

Q12) Conditional event mapping $(\cdot|\cdot\cdot)$ can be extended in a well-defined way to $(\cdot|\cdot\cdot)$: (range of $(\cdot|\cdot\cdot)$ over $B \times B$)$\times$ (range of $(\cdot|\cdot\cdot)$ over $B \times B$) so that, using and reusing the P_0 notation as above, for any probability measure P over b, P_0 and P_{00} are well-defined probability measures over their respective domains, so that for all a, b, c, d in B, provided $P(cd) > 0$,

(i) $P_{00}((a|b)|(c|d)) = P_0((a|b) \cdot (c|d))/P(c|d)$, noting the natural reduction when $b = d$, that $P_{00}((a|b)|(c|b)) = P(a|bc)$;

(ii) A weighted form of the well-known classical logic material conditional property of "import-export" holds as:

$$P_{00}((a|b)|c) = w \cdot P(a|bc) + (1-w) \cdot P(a|b),$$

where w in $[0,1]$ is some weight dependent in general on $P(ab), P(b)$, $P(abc), P(bc), P(c)$, etc.

4. Data Fusion, C3, and Decision-making Using Conditionals.

A basic thrust of the present research is to address the above problem through the development of a comprehensive, mathematically sound, and computationally feasible, way to qualify and quantify conditional or causal relations, compatible with all conditional probability evaluations under the basic assumption - now generally refered to as the Stalnaker Thesis [39] - that the above equation can be satisfied by some appropriate constructible choice of entities outside of the original boolean algebra of events.

This development is expected to lead to an intrinsic conditional event-forming extension of the current standard approach to rational decision-making. The latter, at present, utilizes the concept of a (joint) sigma algebra of -in effect- unconditional events as its basis for all observed data, all parameter vectors of interest, as well as for all actions or decisions, with the requisite conditional relations among these variables expressible only in numerical form, not syntactically, as the proposed extension would do. This is part of a more long-range view of deriving a unified approach to the combining of disparate information, relative to the overall data fusion and C3 (command, control, communications) problem. (See [11, 12, 23, 13, 18, 19], [24], sect V, for specific applications to these problems). Some of this work entails extending conditioning to a fuzzy set framework [3, 20], utilizing this author and Prof. H.T. Nguyen's techniques of representing fuzzy sets via nested random sets [17]. Such information can be expected to be in either of the following possibly overlapping categories, arriving from many differing sources: unconditional or conditional; linguistic-narrative or stochastic-random; and indicative-declarative or modal/temporal in the extended sense, including not only possibility or necessity, but also manifesting the opinion of an agent, such as belief, knowledge, emotion, among other possible illocutionary factors. Finally, note, apropos to the comments at the end of Q11), in addition to the preservation of logical operations on conditionals by such conditional rv, it is just as important to be able to extend also the highly non-boolean ordinary arithmetic operations to conditional form. In a similar direction, possible tie-ins with conditional qualitative probability structures (as shown in the case of GNW for the interval of events approach, as mentioned previously) must also be perused for full solutions to the Basic Problem.

5. Interval of Events/Boolean-like Operator Approach to Conditioning.

In the evolving of a theory of conditional forms, as indicated above, two basic philosophies of approach have naturally arisen: 1) the interval of events/boolean-like operator approach - and its mathematical equivalent forms, including the principal ideal coset, ordered pairs under a natural equivalence, and three-valued set indicator function forms; and 2) the product space approach. The first approach is historically the older, with a number of individuals sharing at least this assumed form for conditional events $(a|b)$ representing "if b, then a", though differing on the particular boolean-like algebra to be assigned. In brief, it can be shown that for properties Q1), Q2), Q3), and for Q6), Q7) somewhat strengthened, so that, for all a, b, c, d in boolean algebra B, $(a|b) = (c|d)$ implies $b = d$ and $(a|b) = (c|b)$ implies $ab = cb$, then the following equivalent forms must hold up to a global isomorphism (see e.g. [14], Chapt. 12 et passim):

$$(a|b) = [ab, b' \vee ab] = \{x \text{ in } B : ab \leq x \leq b' \vee ab\} = B \cdot b' \vee ab = \{yb' \vee ab : y \text{ in } B\}$$

where the first form is called the interval of events form and the second, the principal ideal coset form - the latter, since each $(a|b)$ is in principal ideal boolean quotient algebra $B/B \cdot b'$. The above is also equivalent to the three-valued logic form - using the standard Stone Representation -

$$d(a|b)(x) = 1, \text{ if } x \text{ in } ab; \quad d(a|b)(x) = 0, \text{ if } x \text{ in } a'b; \quad d(a|b)(x) = u, \text{ if } x \text{ in } b'$$

where u is a third value (indeterminate) between 0 and 1. (See [14], sect.1.3.)

We call an operator F among conditional events $(a_i|b_i)$ boolean-like, if there are actually ordinary boolean operators $f = f(a_i, b_i, i = 1, \ldots, n)$, $g = g(a_i, b_i, i = 1, \ldots, n)$ such that $F((a_i|b_i),\ i = 1, \ldots, n) = (f|g)$. In the case of the latter, two pre-eminent boolean-like, or extended, boolean algebras have been proposed to be defined among conditional events of the interval or coset form: GNW (Goodman-Nguyen-Walker) and SAC (Schay-Adams-Calabrese), each having certain advantages and disadvantages., (See [14], sect. 3.5, [2], [24], sects. I,B1-3 for explicit formulas for these algebras.)

For example, GNW: is derivable as the natural functional image extension of the ordinary unconditional boolean operations; produces that unique (distributive and deMorgan) lattice order among conditional events which is fully compatible with the numerical ordering of all conditional probabilities; is that unique Stone algebra among all possible boolean-like conditional event algebras; has a full Stone representation, extending the standard unconditional relations; is related directly to a conditional qualitative probability structure; has a relatively simple normal form expansion, extending the usual boolean canonical expansion; yields essentially the underlying algebraic structure for the newly-emerging concept of "rough sets";and, via the three-valued indicator functional, is isomorphic to Lukasiewicz' very well- investigated and justified three-valued logic. (See [26], [14], [22], [24], [21], [25], [9].)

On the other hand, SAC disjunction can be related to interval algebra intersection ([24], sect IB3) and all of SAC is isomorphic (via also the three-valued indicator functional) to Sobocinski's three-valued logic ([14], sect. 3.5), which is an alternative natural choice for uses in combining undefined quantities with others. (Again, see [2] for further properties of SAC.) Both the last result and the characterization of GNW are special cases of a far-reaching theorem connecting isomorphically all choices of three-valued truth-functional logics with all choices of boolean-like algebras over conditionals represented as intervals of events ([14], sect. 3.4; [25], sect. D, Theorem 6). In addition, it has been recently pointed out that SAC disjunction and conjunction are natural weighted averaging operators and can play an important role in modeling of disjoint-like information (see [21], sect. 3).

6. Deficiencies of the Interval of Events Approach.

Despite the above positive results, GNW, SAC, and all other boolean-like algebras relative to intervals of events as conditionals, fail to satisfy a number of key criteria necessary to the development of a fully satisfactory theory of conditional decisions. These include: 1) inability to have a full boolean structure (GNW being the strongest, having a deMorgan Stone algebra form, while SAC is a deMorgan non-distributive non-full lattice, though it is a separate semi-lattice with respect to conjunction and disjunction); 2) lack of a sound definition for nested or higher order conditionals, with the consistency relation satisfied $(P((a|b)|(c|d)) = P((a|b) \cdot (c|d))/P(c|d))$ - though NGUYEN [35] and GEHRKE and WALKER

[8], among others, have developed higher order conditional events within a Stone algebra setting, at the expense of the above identity; 3) lack of a basis for developing random conditional events and conditional random variables, so that it would be meaningful to have a decision theory based upon conditionals, and in particular, the relation $((Y|Z)-1)(a \times b|R \times b) = ((Y-1)(a)|(Z-1)(b))$, for any rv Y and Z and real Borel events a, b; and 4) not being reducible to the usual probability independence relations, i.e., we need to have $P((a|b) \cdot (c|d)) = P(a|b) \cdot P(c|d)$, when ab, b and cd,d are pairwise P-independent. (See [24], sect.II; [21], sect. 5 for additional details.)

In summary, both GNW and SAC satisfy properties Q1) - Q7), with only GNW satisfying Q8), and despite some elaborate constructions (again, see [14], sects. 5.3, 8.1), neither conditional event algebra satisfies any of Q9) - Q12). This is, basically, because the range space of conditionals, under the interval of events approach using boolean-like operations is always non-boolean. On the other hand, the interval of events approach to conditioning has brought forth much renewed interest in the subject, as can be seen by the publication of the monograph [16], where thirteen researchers contributed new material, and by the convening of the First CEAPL (Conditional Event Algebra and Conditional Probability Logic) Workshop, March 13- 14, 1992, at NRaD, San Diego, where over twenty researchers attended.

7. Product Space Approach to Conditionals.

The product space approach has been developed to address the above problems of the interval of events approach. First, via a suggestion of D.Bamber, NRaD (personal communication), conditional probabilities were seen to satisfy always the identity

$$P(a|b) = \sum_{j=0}^{\infty} P(a) \cdot (P(b'))^j$$

which, in turn, has the obvious product space interpretation through two-stage coin tossing: $P(a|b)$ = probability of eventually obtaining successfully an a from the first coin, where failure means the occurrence of a', following a successful b occurrence from the second coin, where failure - the occurrence of b' - does not permit a toss of the first coin. This led to the formulation that any conditional event could be alternatively expressed as the infinite disjoint disjunction of cylinder events:

$$(a|b) = ab \times L \times L \times \ldots \vee b' \times ab \times L \times L \times \ldots \vee b' \times b' \times ab \times L \times L \times \ldots \vee \ldots .$$

Here, L is the universal element of the boolean algebra containing a, b, c $, d, \ldots$, and where, if P_0 is the usual countably infinite factor product probability measure formed from P as its common marginal factor probability. Then, as a check, it is easily seen that , for $P(b) > 0$, $P_0((a|b)) = P(a|b)$.

All of the above can be put in the context of first forming from a given probability space (L, B, P) the countably infinite product space with identical marginal components. Since one can easily obtain the recursive relation

$$(a|b) = ab \vee (b' \times (a|b))$$

from the first equation, all desired logical combinations of these conditionals can likewise be recursively obtained, yielding more complex, but fully computable counterparts to GNW and SAC, determined uniquely from the product space structure, not by any other criteria. For example, one can show using this technique

$$(a|b) \cdot (c|d) = ((abcd \vee abd' \times (c|d) \vee (cdb' \times (a|b))|b \vee d),$$

where we extend the definition of a conditional event $(a|b)$ in L_0, as in the first equation, to say $(Q|b)$ also in L_0, by formal replacement of consequent ab by Q. Also, a number of explicit formulas and relations involving the product space approach have been obtained, Most importantly, the product space approach answers in the affirmative all of the deficiencies produced by the interval of events approach, i.e., properties Q1) - Q12) are all satisfied! (See [27];[22], sect. 3; [24], sect. IV; [21], sects. 6, 7.)

8. Further Properties of the Product Space Approach.

In place of the common GNW and SAC "import-export" property of nested conditionals,

$$P_0((a|b)|c) = P(a|bc),$$

for all a, b, c in B with $P(bc) > 0$, the product space approach yields rather the weighted form

$$P_0((a|b)|c) = P(b|c) \cdot P(a|bc) + P(b'|c) \cdot P(a|b),$$

among other desirable properties. In fact it has just been shown that any reasonable model for conditional events cannot satisfy both traditional import-export as well as have consistent nested conditionals (in the sense described above). (See [29].) It is also important to point out that the product space produces essentially the same algebraic properties that McGee has obtained via his rational betting/coherency approach to conditioning (though McGee never obtained explicit forms for the conditionals themselves). (See [34].) Even more significantly, Bas Van Fraasen's "Stalnaker Bernoulli models" independently proposed in 1976 and bucking the David Lewis triviality scare in logic (as discussed in a previous section here) [40], coincide with the product space approach. But, apparently, Van Fraasen has not gone on to develop these ideas much further, although he has defined a concept of conditioning, extending that of this author which yields closure relative to the entire product space B_0, derived from the initial boolean algebra of unconditional events B. It should be noted that a recent characterization has been discovered which shows that essentially the only conditional event algebra which has the strong independence property

$$P_0(e \cdot b'd' \cdots (a|b) \cdot (c|d) \cdots) = P(eb'd' \cdots) \cdot P_0((a|b) \cdot (c|d) \cdots),$$

for all probability measures P over B and all $a, b, c, d, e, \ldots$ in B, must be the product space approach ([22], theorem 18). (See [25], sect. H, for other characterizations of the product space approach.)

In summary, while the product space approach obviously introduces more complex computations than the interval of events approach as ,e.g., through SAC and GNW algebras, it is theoretically more sound. Hence a basic tradeoff exists between the use of the two schools of approach ([24], sect. IVD). (For further tie-ins between the two approaches, where in fact GNW and SAC, except for zero- and unity type conditional events, are seen to be imbeddable in the product space setting, see [25], sect. I.)

9. Deduction, Powerdomain Relations, and General Issues.

A major issue arising naturally from the modeling of inference rules, or more generally, conditionals is that of the associated partial or lattice (or even pre-) order generated by the choice of algebra used in their representation. This type of order is naturally interpreted as the basic deducing or causality relation among the inference rules or conditionals. It has already been seen that among all reasonable entities that we could call conditionals compatible with conditional probabilities as shown in properties Q7),Q7′),Q8),Q8′), essentially there is only one type of such order relation among conditionals. Consequently, all questions concerning deducts among just conditionals can be - at least in theory - handled in a straightforward way. But, it is obvious from the explicit forms that conjunctions and disjunctions take in the product space solution to the Basic Problem, closure does not hold in general for these operations relative to being in simple conditional form - although closure does hold for all of these operations relative to the parent product space B_0. (See [24], sect. IV,C.)

Thus, the issue of characterizing the deductive closure - i.e., logical closure of the set of deduct- of a given set of conditionals is no longer the simple task as before. All of the above can be couched very naturally in powerdomain terminology (see e.g. GUNTHER, [30] for background), where we wish to characterize as simply as possible the following typical deduction problems:

1. The deductive closure A^*, relative to the entire product space algebra, of any given set A of conditionals of interest, i.e.,

$$A^* = \text{comb}\,(\cdot, \vee)(\text{ded}\,(A)),$$

where ded(A) is the deduction class of A, or, equivalently, the upper powerdomain of $A, A^{\#}$, i.e., the maximal class (via subclass inclusion) of full deducts of A, i.e., for each $(a|b)$ in $A^{\#}$, there is always some $(c|d)$ in A such that $(c|d)$ deduces $(a|b) : (c|d) \leq (a|b)$.

2. Given any collections of conditionals A, B from B_0 and any subcollections A_1 from comb$(\cdot, \vee)(A)$ and B_1 from comb$(\cdot, \vee)(B)$, find characterizing criteria when:

(i) B_1 fully deduces A_1, i.e., when $A_1 >^{\#} B_1$, i.e., for each element in A_1, there is some element in B_1 which deduces the former (or the former is $\geq$ the latter).

(ii) B_1 efficiently deduces or causes A_1, i.e., when $A_1 \tilde{>} B_1$, i.e., for each element in B_1, its deduction class always intersects with A_1.

3. One can similarly couch in elementary powerdomain terms, full and efficient deductions, mixes of deductions, sandwiches of deductions, etc. The main point here is that all questions of deduction among conditional events with antecedents and consequents from B and their naturally generated logical combinations- in general, not conditional events, but more complex forms - all lying in the same product space B_0, can be immediately translated into powerdomain issues, specialized to this situation. The product space approach taken here is expected to provide tie-ins with current non-monotonic logic systems, hopefully improving upon the empirical/hybrid approaches of DUBOIS and PRADE [7] and CALABRESE [3] in the use of conditionals.

4. What specific conditional event deduction and related concepts can be abstracted to the general powerdomain level, and perhaps contribute new insights? Since the product space approach remains boolean relative to the original underlying space of unconditional events, unlike the previous approaches, the process of conditioning can now be thought of as a special case of developing a particular type of functor from the standard category Set back into Set. On the other hand, it is not too difficult to decompose the entire negation b' of the antecedent of any conditional event $(a|b)$ - corresponding to the intermediate value of the three-valued logic form of the interval of events approach into the union $(b' \times ab) \vee (b' \times b' \times ab) \vee \ldots$ $(= b' \times (a|b))$ of those subevents contributing to the occurrence and the union $(b' \times a'b) \vee (b' \times b' \times a'b) \vee \ldots (= b' \times (a|b))$ of those contributing to the non occurrence of the conditional event. The correspondence between these two component parts and the entire negation characterizes the relation between the two approaches, and somehow, should be reflected in the deeper category theory setting above; in view of the previously-mentioned characterization theorem for all possible truth-functional three-valued logics, this possibly may shed new light on the relation between all such logics and classical logic.

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REFERENCES

[1] Calabrese, P.G. (1975): The probability that p implies q, abstract in *American Mathematical Society Notices*, vol. 22, no. 3, A430-431.

[2] Calabrese, P.G. (1987): An algebraic synthesis of the foundations of logic and probability, *Information Sciences*, vol. 42, 187-237.

[3] Calabrese, P.G.: A theory of conditional information with applications, *IEEE Trans. Systems, Man, & Cybernetics*, to appear.

[4] Calabrese, P.G.: Simpson's Paradox and the algebras of conditional events, to be submitted for publication.

[5] Copeland, A.H. (1950): Implicative boolean algebras, *Math. Zeitschrift*, vol 53, no. 3, 285-290.

[6] Copeland, A.H. and F. Harary (1953): A characteristic of implicative boolean algebras, *Canadian Journal of Math.*, vol. 5, 465-469.

[7] Dubois, D. and Prade, H. (1991): Conditional objects and nonmonotonic reasoning, *Proceedings Second Intern. Conf. Principles of Knowledge Rep. & Reasoning*, (J.Allen, R. Fikjes, & E. Sandewall, eds.), Cambridge, MA., April, 175-185.

[8] Gehrke, M. and Walker, E.A.: Iterating conditionals and symmetric Stone algebras, *IEEE Trans. Systems, Man, & Cybernetics*, to appear.

[9] Gehrke, M. and Walker, E.A.: On the structure of rough sets, to be submitted.

[10] Goodman, I.R. and Nguyen, H.T. (1985): *Uncertainty Models for Knowledge-Based Systems*, North-Holland, Amsterdam.

[11] Goodman, I.R. (1988): Applications of a conditional event algebra to data fusion, *Proceedings 1988 Tri-service data fusion Symposium*, Applied Physics Lab., Laurel, MD, May, 179-187.

[12] Goodman, I.R. (1989): Applications of a conditional event algebra to data fusion, *Proceedings 1989 Tri-service Data Fusion Symposium*, May, 181-193.

[13] Goodman, I.R. (1991): Development and application of some new procedures to the modeling and combining of linguistic and probabilistic information for C3 systems, *Proceedings 1991 Symposium C2 research, Nat'l. Def. Univ.*, Wash., D.C., June, 178-190.

[14] Goodman, I.R. and H.T. Nguyen (1991): *Conditional Inference and Logic for Intelligent Systems: A theory of Measure-Free Conditioning*, North-Holland, Amsterdam.

[15] Goodman, I.R. (1991): Evaluations of combinations of conditional information I: a history, *Information Sciences*, vol 42, no. 1, Sept.-Dec., 79-110.

[16] Goodman,I.R., Gupta, M.M., Nguyen,H.T. and Rogers, G.S. (1991): (eds.), *Conditional Logic in Expert Systems*, North-Holland, Amsterdam.

[17] Goodman, I.R. (1991): Algebraic and probabilistic bases for fuzzy sets and the development of fuzzy conditioning, in monograph *Conditional Logic in Expert Systems* (I.R. Goodman, M.M. Gupta, H.T. Nguyen, & G.S. Rogers, eds.), North-Holland, Amsterdam, 1-69.

[18] Goodman, I.R. (1992): Modeling and fusing of linguistic, probabilistic, and conditional and unconditional information, Specially invited tutorial,*Proceedings Joint Service Symposium on Data Fusion*, vol. III, sect. 3, Applied Physics Lab., Laurel, MD, April.

[19] Goodman, I.R. (1992): The role of conditional event algebra in the modeling of C3 systems, *Proceedings 1992 Symposium C2 Research*, Naval Post.School, Monterey, CA, June, 191-205.

[20] Goodman, I.R. (1993): Conditional events and fuzzy conditional events viewed from a product space perspective, *Proceedings Fifth Intern. Fuzzy Systems Assoc. World Congress*, Seoul, Korea, July, to appear.

[21] Goodman, I.R. (1993): Conditional event algebra: a new approach, *Proceedings Second Intern. Symp. on Uncertainty Modeling & Analysis (ISUMA '93)*, Univ. of Md., College park, MD, April , 55-62.

[22] Goodman, I.R. (1993): Toward a more comprehensive theory of conditional events and conditional event algebra, *Proceedings 1993 Intern. Joint Conf. on AI (IJCAI), Workshop on Conditionals in Knowledge Representation*, Chambery, France, Aug.30.

[23] Goodman, I.R.: A formal theory of C3 and data fusion, chapter in book *C3 Advanced Concepts & Paradigms: The JDL Research Program* (C.R. Jones, ed.), Amer. Inst. Aero. & Astro. Press, Wash., D.C., to appear.

[24] Goodman, I.R.: Toward a comprehensive theory of linguistic and probabilistic evidence: Two new approaches to conditional event algebra, *IEEE Trans. Systems, Man, & Cybernetics*, to appear.

[25] Goodman, I.R.: A theory of conditional information for probabilistic inference in intelligent systems: III, mathematical appendix, *Information Sciences*, to appear.

[26] Goodman, I.R. and Nguyen, H.T.: A theory of conditional information for probabilistic inference in intelligent systems: I, interval of events approach, in *Advances in Fuzzy Theory & Technology*, Vol. 1 (P. Wang, ed.), Bookwright Press, Rayleigh-Durham, NC, to appear.

[27] Goodman, I.R.: A theory of conditional information for probabilistic inference in intelligent systems: I, product space approach, *Information Sciences*, to appear.

[28] Goodman, I.R.: Applications of conditional event algebra to some problems in expert systems, to be submitted for publication.

[29] Goodman, I.R.: The import-export property and nested conditioning of conditional events, to be submitted.

[30] Gunter, C.A. (1992): Powerdomains, conditional event algebras, and their application in the semantics of programming languages, Final Tech. Report, Univ. of PA, Dept. of Computer & Information Sciences, March 25.

[31] Harper,W.L., Stalnaker,R. and Pearce, G. (1981): *Ifs: Conditionals, Belief, Decisions, Chance, and Time*, D. Reidel, Dordrecht, Holland.

[32] Jackson, F. (1991): (ed.), *Conditionals*, Oxford Univ. Press.

[33] Lewis, D. (1976): Probabilities of conditionals and conditional probabilities, *Philosophical Review*, vol. 85, 297-315.

[34] McGee, V. (1989): Conditional probabilities and compounds of conditionals, *Philosophical Review*, vol. 4, 485–541.

[35] Nguyen, H.T.: Intervals in boolean rings: approximation and logic, *Journ. Foundations Computer & Decision Sciences*, to appear.

[36] Nguyen, H.T. and Goodman, I.R.: On modeling of if-then rules for probabilistic inference, *Journ. Intelligent Systems*, to appear.

[37] Popper, K.R. (1963): *Conjectures and Refutations*, Routledge and Kegan Paul, London, p. 390, formula 2.

[38] Rescher, N. (1969): *Many-valued Logics*, McGraw-Hill, New York.

[39] Stalnaker, R.C. (1970): Probability and conditionals, *Philosophy of Science*, vol. 37, 64-80.

[40] Van Fraasen, B. (1976): Probabilities of conditionals, in monograph *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science* (W.L. Harper & C.A. Hooker, eds.), D. Reidel, Dordrecht, Holland, 261-300.